

OPEN SETS OF EXPONENTIALLY MIXING ANOSOV FLOWS

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ABSTRACT. If a flow is sufficiently close to a volume-preserving Anosov flow and $\dim \mathbb{E}_s = 1$, $\dim \mathbb{E}_u \geq 2$ then the flow mixes exponentially whenever the stable and unstable foliations are not jointly integrable (similarly if the requirements on stable and unstable bundle are reversed). This implies the existence of non-empty open sets of exponentially mixing Anosov flows. As part of the proof of this result we show that \mathcal{C}^{1+} uniformly expanding suspension semiflows (in any dimension) mix exponentially when the return time is not cohomologous to a piecewise constant.

1. INTRODUCTION & RESULTS

Anosov flows [1] are arguably the canonical examples of chaotic dynamical systems and the rate of mixing (decay of correlation) is one of the most important statistical properties. Nevertheless our knowledge of the rate of mixing of Anosov flows remains unsatisfactory.

Let $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$ be an Anosov flow on \mathcal{M} , a smooth compact connected Riemannian manifold. There exists a ϕ^t -invariant continuous splitting of tangent space $T\mathcal{M} = \mathbb{E}_s \oplus \mathbb{E}_0 \oplus \mathbb{E}_u$ where \mathbb{E}_0 is the line bundle tangent to the flow, \mathbb{E}_s is the stable bundle in which there is exponential contraction and \mathbb{E}_u is the unstable bundle in which there is exponential expansion. It is known that each Anosov flow admits a unique SRB measure which will be denoted μ . The focus of this text is to prove exponential mixing with respect to the SRB measure. By this we mean the existence of $C, \gamma > 0$ such that $|\int_{\mathcal{M}} f \cdot g \circ \phi^t d\mu - \int_{\mathcal{M}} f d\mu \int_{\mathcal{M}} g d\mu| \leq C \|f\|_{\mathcal{C}^1} \|g\|_{\mathcal{C}^1} e^{-\gamma t}$ for all $f, g \in \mathcal{C}^1(\mathcal{M}, \mathbb{R})$ and for all $t \geq 0$. (An approximation argument means that exponential mixing for \mathcal{C}^1 observables implies also exponential mixing for Hölder observables.) In the following we will use the expression *mixes exponentially* to mean with respect to the unique SRB measure for the flow.

Dolgopyat [12], building on work by Chernov [11], showed that Anosov flows with \mathcal{C}^1 stable and unstable foliations mix exponentially whenever the stable and unstable foliations are not jointly integrable. In particular this means that geodesic flows on surfaces of negative curvature mix exponentially. A question of foremost importance is to show that statistical properties hold for an open and dense set of systems. The problem here is that the requirement of regularity for both foliations simultaneously is not typically satisfied for Anosov flows [15].

Liverani [17] showed that all contact (with \mathcal{C}^2 contact form) Anosov flows mix exponentially with no requirement on the regularity of the stable and unstable foliations. Liverani's requirement of a \mathcal{C}^2 contact form has two important consequences. Firstly it guarantees that $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable and this is a property which is

Date: 13th September 2016.

2010 Mathematics Subject Classification. Primary: 37A25; Secondary: 37C30.

Key words and phrases. Anosov flow, Mixing rates.

With pleasure we thank Ian Melbourne, Masato Tsujii and Sina Türelı for stimulating discussions. We are grateful to the ESI (Vienna) for hospitality during the event "Mixing Flows and Averaging Methods" where this work was initiated.

robust under perturbation. Secondly the smoothness of the contact form guarantees the smoothness of the subbundle $\mathbb{E}_s \oplus \mathbb{E}_u$ and the smoothness of the temporal function [17, Figure 2]. This smoothness is essential to Liverani's argument. Unfortunately the existence of a \mathcal{C}^2 contact form cannot be expected to be preserved by perturbations (the consequences of the existence of a smooth contact structure would contradict the prevalence of foliations with bad regularity which was mentioned above). Nevertheless it is hoped that the set of exponentially mixing Anosov flows contains an open and dense set of all Anosov flows.

Tsujii [19] has recently demonstrated the existence of an open and dense subset of volume-preserving three-dimensional Anosov flows which mix exponentially.¹ In this present text we demonstrate that in higher dimensions this question can, to some extent, be answered rather more easily by combining essentially known ideas. A major part of the present text will be devoted to showing that typical \mathcal{C}^{1+} uniformly expanding suspension semiflows mix exponentially.² This is also a new result and of independent interest.

Theorem 1. *Suppose that $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$ is a transitive \mathcal{C}^{1+} Anosov flow and that the stable foliation is \mathcal{C}^{1+} . Then, either the stable and unstable foliations are jointly integrable, or ϕ^t mixes exponentially with respect to the unique SRB measure.*

This result improves the result of Dolgopyat [12] since regularity is only required for the stable foliation whereas in the cited work regularity was required of both foliations. This improvement is a significant advantage in finding open sets of exponentially mixing flows as illustrated by the following theorem.

Theorem 2. *Suppose that $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$ is a \mathcal{C}^{2+} volume-preserving Anosov flow and that $\dim \mathbb{E}_s = 1$ and $\dim \mathbb{E}_u \geq 2$. There exists a \mathcal{C}^1 -neighbourhood of this flow, such that, for all \mathcal{C}^{2+} Anosov flows in the neighbourhood, either the stable and unstable foliations are jointly integrable, or ϕ^t mixes exponentially with respect to the unique SRB measure.*

Since the set of Anosov flows where the stable and unstable bundles are not jointly integrable is \mathcal{C}^1 -open and \mathcal{C}^r -dense in the set of all Anosov flows [13] (see references within concerning the prior work of Brin) the above theorem implies a wealth of open sets of exponentially mixing Anosov flows. To the best of our knowledge, this is the first proof of the existence of open sets of Anosov flows which mix exponentially. Similarly the set of Anosov flows where the stable and unstable bundles are not jointly integrable is \mathcal{C}^1 -open and \mathcal{C}^r -dense in the set of volume preserving Anosov flows.³ This means that an open and dense subset of the volume-preserving Anosov flows such that $\dim \mathbb{E}_s = 1$ and $\dim \mathbb{E}_u \geq 2$ mix exponentially. The conditions of \mathbb{E}_s and \mathbb{E}_u can be reversed and the results remain valid. Consequently a \mathcal{C}^1 -open and \mathcal{C}^r -dense subset of four-dimensional volume-preserving flows $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$ mix exponentially. The proof of Theorem 2 requires the flow to be transitive however, due to Verjovsky [20], codimension one Anosov flows (i.e., $\dim \mathbb{E}_s = 1$ or $\dim \mathbb{E}_u = 1$) on higher dimensional manifolds ($\dim \mathcal{M} > 3$) are transitive and so transitivity

¹Interestingly the set Tsujii constructs doesn't contain the flows which preserve a \mathcal{C}^2 contact form.

²For any $k \in \mathbb{N}$, the notation \mathcal{C}^{k+} means $\mathcal{C}^{k+\alpha}$ for some $\alpha \in (0, 1]$.

³Consider a volume-preserving Anosov flow and assume that $\mathbb{E}_s \oplus \mathbb{E}_u$ is integrable. There exists a section such that the flow can be described as a suspension with constant return time. We will perturb the flow by smoothly modifying the magnitude of the associated vector field in a small ball. Following [13] we can do this in such a way to guarantee that, for the perturbed flow, $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable. Note that the perturbed system is still Anosov and as smooth as before. Since we only changed the magnitude of the vector field the cross-section remains a cross-section and the return map also remains unchanged. Consequently we ensure that the perturbed flow also preserves a smooth volume.

is automatic⁴ in the case of Theorem 2. Section 2 contains the proof of Theorem 1 and the details of how Theorem 2 is derived from it. The proof of the first result rests heavily on the result (Theorem 3 below) concerning exponential mixing for \mathcal{C}^{1+} expanding semiflows. We observe that the ideas in this text are very much limited to the argument presented here and will not suffice to fully answer the question of when in general Anosov flows mix exponentially. For this progress we hope that the work of Dolgopyat [12], Liverani [17], Baladi & Vallée [7] and Tsujii [19] (among others) can eventually be extended and improved.

We proceed by defining the class of \mathcal{C}^{1+} expanding semiflows. Let X be a compact open subset of \mathbb{R}^d and $\alpha \in (0, 1)$. Let $T : X \rightarrow X$ denote a *uniformly expanding \mathcal{C}^{1+} Markov map*. By this we mean that there exists \mathcal{P} , a finite partition into connected open sets of a full measure subset of X such that T is a \mathcal{C}^1 diffeomorphism between ω and $T\omega$ for each $\omega \in \mathcal{P}$. We require that the boundary of each ω to be a finite union of $\mathcal{C}^{1+\alpha}$ -submanifolds. We require that there exists $C_1 > 0$, $\lambda > 0$ such that

$$(1) \quad \|DT^{-n}(x)\| \leq C_1 e^{-\lambda n} \quad \text{for all } x \in X, n \in \mathbb{N},$$

and there exists $C_2 > 0$ such that

$$(2) \quad \left| \ln \frac{\det(DT(x))}{\det(DT(y))} \right| \leq C_2 d(Tx, Ty)^\alpha \quad \text{for all } \omega \in \mathcal{P}, \text{ for all } x, y \in \omega$$

We also require T to be covering in the sense that for every open ball $B \subset X$ there exists $n \in \mathbb{N}$ such that $T^n B = X$. For such maps it is known that there exists a unique T -invariant probability measure absolutely continuous with respect to Lebesgue. We denote this measure by ν . The density of the measure is Hölder (on each partition element) and bounded away from zero. Let $\tau : X \rightarrow \mathbb{R}_+$ denote the *return time function*. We require that τ is $\mathcal{C}^{1+\alpha}$, that there exists $C_3 > 0$ such that

$$(3) \quad \|D\tau(x)DT^{-1}(x)\| \leq C_3 \quad \text{for all } x \in \omega, \omega \in \mathcal{P},$$

and that there exists $C_4 > 0$ such that

$$(4) \quad \tau(x) \leq C_4 \quad \text{for all } x \in \omega, \omega \in \mathcal{P},$$

The suspension semiflow $T_t : X_\tau \rightarrow X_\tau$ is defined as usual, $X_\tau := \{(x, u) : x \in X, 0 \leq u < \tau(x)\}$ and $T_t : (x, u) \mapsto (x, u + t)$ modulo the identifications $(x, \tau) \sim (Tx, 0)$. The unique absolutely continuous T_t -invariant probability measure⁵ is denoted by ν_τ .

Baladi and Vallée [7] showed that semiflows as above, but with the \mathcal{C}^2 version of assumptions, typically mix exponentially when X is one dimensional. The same argument was shown to hold by Avila, Gouëzel & Yoccoz [6], again in the \mathcal{C}^2 case, irrespective of the dimension of X . Recently Araújo & Melbourne [3] showed that the argument still holds in the \mathcal{C}^{1+} case when X is one dimensional. This weight of evidence means that the following result is not unexpected.

Theorem 3. *Suppose that $T_t : X_\tau \rightarrow X_\tau$ is a uniformly expanding \mathcal{C}^{1+} suspension semiflow as above. Then either τ is cohomologous to a piecewise constant function or there exists $C, \gamma > 0$ such that, for all $f, g \in \mathcal{C}^1(X_\tau, \mathbb{R})$,*

$$\left| \int_{X_\tau} f \cdot g \circ T_t d\nu_\tau - \int_{X_\tau} f d\nu_\tau \int_{X_\tau} g d\nu_\tau \right| \leq C \|f\|_{\mathcal{C}^1} \|g\|_{\mathcal{C}^1} e^{-\gamma t} \quad \text{for all } t \geq 0.$$

⁴In the case where both the stable and unstable bundles are at least 2 dimensional there are examples of non-transitive Anosov flows [14]. Also, as it is remarked in [14], the three dimensional case, where Verjovsky's proof does not work, this question of transitivity of Anosov flows remains open.

⁵ $\nu_\tau(f) = \frac{1}{\nu(\tau)} \int_X \int_0^{\tau(x)} f(x, u) du d\nu(x)$

The proof of the above is the content of Section 3. The estimate for exponential mixing relies on estimates of the norm of the twisted transfer operator given in Proposition 16.

2. ANOSOV FLOWS

This section is devoted to the proof of Theorem 1 and Theorem 2. The proof of Theorem 1 relies crucially on Theorem 3. The proof of Theorem 2 relies crucially on Theorem 1.

2.1. Proof of Theorem 1. Suppose that $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$ is a $\mathcal{C}^{1+\alpha}$ Anosov flow and that the stable foliation is $\mathcal{C}^{1+\alpha}$ for some $\alpha > 0$. The proof is based (as per [5, 2, 3]) on quotienting along local stable manifolds and reducing the problem to the study of the corresponding expanding suspension semiflow. We then use the estimate which is given by Theorem 3.

The argument is exactly the one used in [2] for Axiom A flows. The only difference being that some of the estimates are now Hölder and not \mathcal{C}^1 since here we have merely a $\mathcal{C}^{1+\alpha}$ stable foliation whereas in the reference the foliation is \mathcal{C}^2 . Here we summarise the argument.

By Bowen [8], we can find a section which consists of a family of local sections and by modifying slightly these local sections, as was done in [2], we can guarantee that these local sections are $\mathcal{C}^{1+\alpha}$ and foliated by local stable manifolds. The return map is a uniformly hyperbolic Markov map on the family of local sections [8]. Let Y denote the union of the local sections and let $S : Y \rightarrow Y$ and $\tau : Y \rightarrow \mathbb{R}_+$ denote the return map and return time for ϕ^t to this section. Let η denote the unique SRB measure for $S : Y \rightarrow Y$. Note that τ is constant [2, §3] along the local stable manifolds.

We now quotient along the local stable manifolds (within the local sections) letting $\pi : Y \rightarrow X$ denote the quotient map. Consequently we obtain a map $T : X \rightarrow X$ such that $T \circ \pi = \pi \circ S$. Because of the properties of S , the map T is a uniformly expanding Markov map and satisfies the conditions (1), (2), (3) and (4). Therefore, applying Theorem 3, we have that either the suspension semiflow T_t mixes exponentially or τ is cohomologous to a constant function. If τ is cohomologous to a constant function then [2, Lemma 12] the stable and unstable bundles are jointly integrable so for the rest of the proof, we suppose that τ is not cohomologous to a constant function and hence T_t mixes exponentially.

Let ν denote the unique SRB measure for T ($\nu = \pi_* \eta$). To proceed we observe that the measure ν admits a disintegration into conditional measures along local stable manifolds. We observe [10] that there exists a family of conditional measures $\{\nu_x\}_{x \in X}$ (ν_x supported on $\pi^{-1}x$) such that

$$\eta(v) = \int_X \nu_x(v) d\eta(x)$$

for all continuous functions $v : Y \rightarrow \mathbb{R}$. We also know [10, Proposition 6] that this disintegration has good regularity in the sense that $x \mapsto \nu_x(v)$ is Hölder on each partition element and has uniformly bounded Hölder norm for any Hölder $v : Y \rightarrow \mathbb{R}$.

Let Y_τ, S_t, η_τ be defined analogously to X, T_t, ν_τ . Suppose $u, v : Y_\tau \rightarrow \mathbb{R}$ are Hölder continuous functions. Points in Y are denoted by (x, a) which is given by the product representation of Y by X times the local stable manifolds. To prove that S mixes exponentially, it is convenient to write

$$(5) \quad \int_{Y_\tau} u \cdot v \circ S_{2t} d\eta_\tau = \int_{Y_\tau} u \cdot (v \circ S_t - v_t \circ \pi_\tau) \circ S_t d\eta_\tau + \int_{X_\tau} \tilde{u} \cdot v_t \circ T_t d\nu_\tau$$

where $\tilde{u} : X_\tau \rightarrow \mathbb{R}$, $v_t : X_\tau \rightarrow \mathbb{R}$ are defined as

$$\tilde{u}(x, a) := \int_{\pi^{-1}x} u(y, a) d\nu_x(y), \quad v_t(x, a) := \int_{\pi^{-1}x} v \circ S_t(y, a) d\nu_x(y).$$

The new observables \tilde{u} and v_t are \mathcal{C}^α on each partition element as observed above. To estimate the first term of (5) we observe that

$$(v \circ S_t - v_t \circ \pi_\tau)(y, u) = \int_{\pi^{-1}(\pi y)} v \circ S_t(y, u) - v \circ S_t(z, u) d\nu_{\pi y}(z).$$

Consequently the function v_t is exponentially close to $v \circ S_t$ on each local stable manifold and so

$$(6) \quad \left| \int_{Y_\tau} u \cdot (v \circ S_t - v_t \circ \pi_\tau) \circ S_t d\eta_\tau \right| \leq C \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\alpha} e^{-\tilde{\gamma}t}$$

where $\tilde{\gamma} > 0$ depends on the contraction rate on the stable bundle.

The second term of (5) is estimated using Theorem 3 which says that T_t mixes exponentially since τ is not cohomologous to a piecewise constant. We have

$$(7) \quad \left| \int_{X_\tau} \tilde{u} \cdot v_t \circ T_t d\nu_\tau - \int_{X_\tau} \tilde{u} d\nu_\tau \cdot \int_{X_\tau} v_t d\nu_\tau \right| \leq C \|\tilde{u}\|_{\mathcal{C}^\alpha} \|v_t\|_{\mathcal{C}^\alpha} e^{-\gamma t}.$$

Using estimates (6) and (7) in (5) gives that the flow $S_t : Y_\tau \rightarrow Y_\tau$ mixes exponentially. This in turn implies that the flow ϕ^t is exponentially mixing.

2.2. Proof of Theorem 2. The proof consists of showing that if ϕ^t is \mathcal{C}^1 -close to a volume preserving flow and that $\dim \mathbb{E}_s = 1$, $\dim \mathbb{E}_u \geq 2$ then the stable foliation is \mathcal{C}^{1+} . We then apply Theorem 1.

We recall that the regularity of the invariant foliation of an Anosov flow is given by Hirsch, Pugh & Shub [16] (see also [4, Theorem 4.12]) under the following *bunching* condition. Suppose that $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$ is a \mathcal{C}^{2+} Anosov flow⁶. If there exists $t, \alpha > 0$ such that

$$(8) \quad \sup_{x \in \mathcal{M}} \|D\phi^t|_{\mathbb{E}_s}(x)\| \|D\phi^t|_{\mathbb{E}_{cu}}^{-1}(x)\| \|D\phi^t|_{\mathbb{E}_{cu}}(x)\|^{1+\alpha} < 1,$$

then the stable foliation is $\mathcal{C}^{1+\alpha}$ ($\mathbb{E}_{cu} = \mathbb{E}_u \oplus \mathbb{E}_0$ and is called the *central unstable sub-bundle*).

Following Plante [18, Remark 1], we observe that, in the case when the Anosov flow is volume preserving, $\dim \mathbb{E}_s = 1$ and $\dim \mathbb{E}_u \geq 2$, then the above bunching condition holds true and consequently that the stable foliation is $\mathcal{C}^{1+\alpha}$ for some $\alpha > 0$. This is because volume-preserving means that the contraction in \mathbb{E}_s must equal the volume expansion in \mathbb{E}_u . Since $\dim \mathbb{E}_u \geq 2$ the maximum expansion in any given direction must be dominated by the contraction. Consequently the stable foliation is $\mathcal{C}^{1+\alpha}$. From its definition the bunching condition (8) is robust under \mathcal{C}^1 perturbations of the Anosov flow.

2.3. Remarks. This argument for the regularity of the stable foliation uses crucially that the unstable bundle has dimension at least 2 whilst the stable bundle has dimension 1. Such an argument is therefore not possible if the Anosov flow is three dimensional (see [18] for a counter example). The corresponding result holds when the requirements on the stable and unstable bundles are swapped. When $\dim \mathbb{E}_s \neq \dim \mathbb{E}_u$ then it is again possible to find open sets such that the bunching condition is satisfied although this will not be possible for all such flows.

⁶This is the only place where the flow is required to be \mathcal{C}^{2+} , everywhere else \mathcal{C}^{1+} suffices.

3. EXPANDING SEMIFLOWS

This section is devoted to the proof of Theorem 3. Recall that the semiflow is a combination of a uniformly expanding map $T : X \rightarrow X$ and return time $\tau : X \rightarrow \mathbb{R}_+$. Let m denote Lebesgue measure on X . We will assume, scaling if required, that the diameter of X is not greater than 1 and that $m(X) \leq 1$. We will also assume that $C_1 = 1$ in assumption (1). Suppose that this is not the case originally, then there exists some iterate such that $C_1 e^{-\lambda n} < 1$. We choose some partition element such that returning to this partition element takes at least n iterates. We take \tilde{X} (which will replace X) to be equal to this partition element and choose for \tilde{T} the first return map to \tilde{X} . The new return time τ is given by the corresponding sum of the return time. There is then a one-to-one correspondence between the new suspension semiflow and the original. It is simply a different choice of coordinates for the flow which has the effect that the expansion per iterate is increased and the return time increases correspondingly. This is not essential but it is convenient because below we can choose a constant cone field which is invariant. We will also assume for notational simplicity that $C_4 \leq 1$, i.e., that $\tau(x) \leq 1$ for all x . This can be done without loss of generality, simply by scaling uniformly in the flow direction. Let $\Lambda > 0$ be such that $\|DT(x)\| \leq e^\Lambda$ for all x . This relates to the maximum possible expansion whereas $\lambda > 0$ relates to the minimum expansion. After these considerations the suspension semiflow is controlled by the constants $\alpha \in (0, 1)$, $\Lambda \geq \lambda > 0$ and $C_2, C_3 > 0$.

Central to the argument of this section are Proposition 6, Proposition 9 and Proposition 16. The first describes how we see, in an exponential way, a key geometric property. The second proposition uses this geometric property and the idea of oscillatory integrals in order to see cancellations on average. The third proposition is the combination of the previous estimates to produce the key estimate on the norm of the twisted operators.

3.1. Basic Estimates. Let $C_5 = 2C_3/(1 - e^{-\lambda})$, let $\tau_n := \sum_{j=0}^{n-1} \tau \circ T^j$ and let \mathcal{P}_n denote the n^{th} refinement of the partition. For convenience we will systematically use the notation $\ell_\omega := T^n|_\omega^{-1}$ for any $n \in \mathbb{N}$, $\omega \in \mathcal{P}_n$. Let $J_n(x) = 1/\det DT^n(x)$.

Lemma 1. $\|D(\tau_n \circ \ell_\omega)(x)\| \leq \frac{1}{2}C_5$ for all $n \in \mathbb{N}$, $\omega \in \mathcal{P}_n$, $x \in T^n\omega$.

Proof. Let $y = \ell_\omega(x)$ and observe that

$$D(\tau_n \circ \ell_\omega)(x) = \sum_{k=0}^{n-1} D\tau(T^k y) D(T^k \circ \ell_\omega)(T^k y)$$

Consequently, using assumptions (1) and (3), $\|D(\tau_n \circ \ell_\omega)\| \leq C_3 \sum_{k=0}^{n-1} e^{-\lambda(n-k)}$. As $\sum_{k=0}^{\infty} e^{-\lambda k} = (1 - e^{-\lambda})^{-1}$ the required estimate holds. \square

Lemma 2. *There exists $C_6 > 0$ such that, for all $n \in \mathbb{N}$, $\omega \in \mathcal{P}_n$*

$$\left| \ln \frac{\det(D\ell_\omega(x))}{\det(D\ell_\omega(y))} \right| \leq C_6 d(x, y)^\alpha \quad \text{for all } x, y \in T^n\omega.$$

Proof. We write $\ell_\omega = g_1 \circ \dots \circ g_n$ where each g_k is the inverse of T restricted to the relevant domain. Let $x_k = T^k \ell_\omega x$, $y_k = T^k \ell_\omega y$. Consequently $\det(D\ell_\omega(x)) = \prod_{k=1}^n \det(Dg_k(x_k))$ and so

$$\left| \ln \frac{\det(D\ell_\omega(x))}{\det(D\ell_\omega(y))} \right| \leq \sum_{k=1}^n \left| \ln \frac{\det(Dg_k(x_k))}{\det(Dg_k(y_k))} \right|$$

Assumption (2) implies that $\left| \ln \frac{\det(Dg_k(x_k))}{\det(Dg_k(y_k))} \right| \leq C_2 d(x_k, y_k)^\alpha$. Using also assumption (1) we obtain a bound $\sum_{k=1}^n C_2 (e^{-\lambda(n-k)})^\alpha d(x, y)^\alpha$. To finish the estimate let $C_6 := C_2 \sum_{j=0}^\infty e^{-\lambda \alpha j}$. \square

Lemma 3. *There exists $C_7 > 0$ such that*

$$\sum_{\omega \in \mathcal{P}_n} \|J_n\|_{L^\infty(\omega)} \leq C_7 \quad \text{for all } n \in \mathbb{N}.$$

Proof. For each $\omega \in \mathcal{P}_n$ there exists $x_\omega \in \omega$ such that $m(\omega) = J_n(x_\omega)m(T^n\omega)$. This choice means that $\sum_{\omega \in \mathcal{P}_n} J_n(x_\omega) \leq m(x) (\inf_\omega m(T^n\omega))^{-1}$. By Lemma 2

$$\|J_n\|_{L^\infty(\omega)} / J_n(x_\omega) \leq e^{C_6}.$$

Consequently $\sum_{\omega \in \mathcal{P}_n} \|J_n\|_{L^\infty(\omega)} \leq C_7$ where $C_7 := e^{C_6} / \inf_\omega m(T^n\omega)$. \square

3.2. Twisted Transfer Operators. For $z \in \mathbb{C}$, the twisted transfer operator $\mathcal{L}_z : L^\infty(X) \rightarrow L^\infty(X)$ is defined as usual and has the formula

$$\mathcal{L}_z^n f = \sum_{\omega \in \mathcal{P}_n} (e^{-z\tau_n} \cdot f \cdot J_n) \circ \ell_\omega \cdot \mathbf{1}_{T^n\omega}.$$

We use the standard notation for the Hölder seminorm $|f|_{\mathcal{C}^\alpha(J)}$ where J is any metric space. I.e., $|f|_{\mathcal{C}^\alpha(J)}$ is the supremum of $C \geq 0$ such that $|f(x) - f(y)| \leq C d(x, y)^\alpha$ for all $x, y \in J$, $x \neq y$. The Hölder norm is defined $\|f\|_{\mathcal{C}^\alpha(J)} := |f|_{\mathcal{C}^\alpha(J)} + \|f\|_{L^\infty(J)}$. We also define a norm which respects the Markov partition \mathcal{P} . Slightly abusing notation, let

$$|f|_{\mathcal{C}^\alpha(X)} := \sup_{\omega \in \mathcal{P}} \sup_{\substack{x, y \in \omega, \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

and let $\|f\|_{\mathcal{C}^\alpha(X)} := |f|_{\mathcal{C}^\alpha(X)} + \|f\|_{L^\infty(X)}$. Let $\mathcal{C}^\alpha(X) := \{f : X \rightarrow \mathbb{R} : |f|_{\mathcal{C}^\alpha(X)} < \infty\}$. This a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{C}^\alpha(X)}$. Define, for all $b \in \mathbb{R}$, the equivalent norm

$$\|f\|_{(b)} := \frac{1}{(1+|b|^\alpha)} |f|_{\mathcal{C}^\alpha(X)} + \|f\|_{\mathcal{C}^\alpha(X)}.$$

Observe that, using Lemma 3, $\|\mathcal{L}_z^n f\|_{L^\infty(X)} \leq C_7 e^{-\Re(z)n} \|f\|_{L^\infty(X)}$ for all $n \in \mathbb{N}$, $f \in L^\infty(X)$.

The argument of this section depends on choosing $\sigma > 0$ sufficiently small. At various points in the following chosen smaller than before if required. At each stage the choice will depend only on the system (x, T, τ) .

Lemma 4. *There exists $C_8 > 0$ such that, for all $z = a + ib$, $a > -\sigma$, $f \in \mathcal{C}^\alpha(X)$, $n \in \mathbb{N}$,*

$$\|\mathcal{L}_z^n f\|_{\mathcal{C}^\alpha(X)} \leq C_8 e^{-(\alpha\lambda - \sigma)n} |f|_{\mathcal{C}^\alpha(X)} + C_8 e^{\sigma n} (1 + |b|^\alpha) \|f\|_{L^\infty(X)}.$$

Proof. Suppose that $\omega \in \mathcal{P}_n$, $f \in \mathcal{C}^\alpha(X)$ and $x, y \in T^n\omega$, $x \neq y$, then

$$(e^{-z\tau_n} \cdot f \cdot J_n)(\ell_\omega x) - (e^{-z\tau_n} \cdot f \cdot J_n)(\ell_\omega y) = A_1 + A_2 + A_3 + A_4$$

where

$$\begin{aligned} A_1 &= (e^{-ib\tau_n(\ell_\omega x)} - e^{-ib\tau_n(\ell_\omega y)})(e^{-a\tau_n} \cdot f \cdot J_n)(\ell_\omega x) \\ A_2 &= e^{-ib\tau_n(\ell_\omega y)}(e^{-a\tau_n(\ell_\omega x)} - e^{-a\tau_n(\ell_\omega y)})(f \cdot J_n)(\ell_\omega x) \\ A_3 &= e^{-z\tau_n(\ell_\omega y)}(f(\ell_\omega x) - f(\ell_\omega y)) \cdot J_n(\ell_\omega x) \\ A_4 &= e^{-z\tau_n(\ell_\omega y)} f(\ell_\omega y)(J_n(\ell_\omega x) - J_n(\ell_\omega y)). \end{aligned}$$

By Lemma 1 $|A_1| \leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega x) 2 \min(|b| \frac{C_5}{2} d(x, y), 1)$. Since $\min(u, 1) \leq u^\alpha$ for all $u \geq 0$, $|A_1| \leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega x) 2 |b|^\alpha (\frac{C_5}{2})^\alpha d(x, y)^\alpha$. Again, by

Lemma 1, $|A_2| \leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega x) |a| \frac{C_5}{2} d(x, y)$. Using assumption (1) $|A_3| \leq (e^{-a\tau_n} \cdot J_n)(\ell_\omega y) e^{-\alpha\lambda n} d(x, y)^\alpha |f|_{\mathcal{C}^\alpha(\omega)}$. Finally, by Lemma 2 $|A_4| \leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega y) C_6 d(x, y)^\alpha$. Summing over $\omega \in \mathcal{P}_n$ we obtain

$$(9) \quad \frac{|\mathcal{L}_z^n f(x) - \mathcal{L}_z^n f(y)|}{d(x, y)^\alpha} \leq \|\mathcal{L}_a^n 1\|_{L^\infty(X)} \left[((2|b|^\alpha + |a|) \frac{C_5}{2} + C_6) \|f\|_{L^\infty(X)} + C_7 e^{-\lambda n} |f|_{\mathcal{C}^\alpha(X)} \right]$$

To finish we observe that $\|\mathcal{L}_z^n f\|_{L^\infty(X)} \leq C_7 e^{\sigma n} \|f\|_{L^\infty(X)}$ and choose C_8 according to the above equation. \square

Lemma 4, observing the definition of the $\|\cdot\|_{(b)}$ norm, implies the following uniform estimate.

Lemma 5. *For all $z = a + ib$, $a > -\sigma$,*

$$\|\mathcal{L}_z^n f\|_{(b)} \leq C_8 e^{\sigma n} \left(e^{-\lambda n} \|f\|_{(b)} + \|f\|_{L^\infty(X)} \right) \quad \text{for all } f \in \mathcal{C}^\alpha(X), n \in \mathbb{N}.$$

3.3. Exponential transversality. The goal of this subsection is to prove Proposition 6 below. This is an extension of Tsujii [19, Theorem 1.4] to the present higher dimensional situation. Much of the argument follows the reasoning of the above mentioned reference with some changes due to the more general setting.

Define the $(d+1)$ -dimensional square matrix $\mathcal{D}^n(x) : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$,

$$\mathcal{D}^n(x) = \begin{pmatrix} DT^n(x) & 0 \\ D\tau_n(x) & 1 \end{pmatrix}.$$

This is notationally convenient since $DT_t(x, s) = \mathcal{D}^n(x)$ whenever $\tau_n(x) \leq s + t < \tau_{n+1}(x)$.⁷ To proceed it is convenient to establish the notion of an invariant unstable cone field. Recall that $C_5 = 2C_3/(1 - e^{-\lambda})$. We define $\mathcal{K} \subset \mathbb{R}^{d+1}$ as

$$\mathcal{K} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a \in \mathbb{R}^d, b \in \mathbb{R}, |b| \leq C_5 |a| \right\}.$$

We refer to \mathcal{K} as a *cone*. We will see now that the width of the cone has been chosen sufficiently wide to guarantee invariance. Note that

$$\begin{pmatrix} DT(x) & 0 \\ D\tau(x) & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} DT(x)a \\ D\tau(x)a + b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

Let $\omega \in \mathcal{P}$ be such that $a = D\ell_\omega(Tx)a'$. Using conditions (1) and (3), we have

$$(10) \quad |b'| = |D\tau(x)a + b| = |D(\tau \circ \ell_\omega)(Tx)a' + b| \leq C_3 |a'| + C_5 e^{-\lambda} |a'| \leq \frac{1}{2} C_5 |a'|.$$

Suppose that $x_1, x_2 \in X$, $n \in \mathbb{N}$ such that $T^n x_1 = T^n x_2$. We write

$$\mathcal{D}^n(x_1)\mathcal{K} \cap \mathcal{D}^n(x_2)\mathcal{K}$$

if $\mathcal{D}^n(x_1)\mathcal{K} \cap \mathcal{D}^n(x_2)\mathcal{K}$ does not contain a d -dimensional linear subspace. In such a case we say that the image cones are *transversal*.

Proposition 6. *Let $T : X \rightarrow X$ be a \mathcal{C}^{1+} uniformly expanding Markov map and $\tau : X \rightarrow \mathbb{R}_+$ as above. Further suppose that there does not exist some $\theta \in \mathcal{C}^1(X, \mathbb{R})$ such that $\tau = \theta \circ T - \theta + \chi$ where χ is constant on each partition element. Then there exists $C_9, \gamma > 0$ such that, for all $y \in X$, $x_0 \in T^{-n}y$,*

$$(11) \quad \sum_{\substack{x \in T^{-n}y \\ \mathcal{D}^n(x)\mathcal{K} \not\cap \mathcal{D}^n(x_0)\mathcal{K}}} J_n(x) \leq C_9 e^{-\gamma n}.$$

⁷If one wished to study the skew-product $G : (x, u) \mapsto (Tx, u - \tau(x))$ this is also the relevant object to study since $\mathcal{D}^n = DG^n$.

The major part of the remainder of this subsection is devoted to the proof of this proposition but first we record a consequence of transversality.

Lemma 7. *Suppose that $\omega, v \in \mathcal{P}_n$, $y \in X$ and that $\mathcal{D}^n(\ell_\omega y)\mathcal{K} \pitchfork \mathcal{D}^n(\ell_v y)\mathcal{K}$. Then there exists $L \subset \mathbb{R}^d$, a 1-dimensional linear subspace, such that, for all $v \in L$*

$$|D(\tau_n \circ \ell_\omega)(y)v - D(\tau_n \circ \ell_v)(y)v| > C_5(|D\ell_\omega(y)v| + |D\ell_v(y)v|).$$

Proof. Let $x_1 = \ell_\omega y$, $x_2 = \ell_v y$. That $\mathcal{D}^n(x_1)\mathcal{K} \pitchfork \mathcal{D}^n(x_2)\mathcal{K}$ means there exists $L \subset \mathbb{R}^d$, a line which passes through the origin, such that, when restricted to the two dimensional subspace $L \times \mathbb{R} \subset \mathbb{R}^{d+1}$, the image cones $\mathcal{D}^n(x_1)\mathcal{K}$ and $\mathcal{D}^n(x_2)\mathcal{K}$ fail to intersect (except at the origin). Observe that

$$\begin{aligned} \mathcal{D}^n(x)\mathcal{K} \cap L \times \mathbb{R} &= \left\{ \begin{pmatrix} DT^n(x)a \\ D\tau_n(x)a+b \end{pmatrix} : |b| \leq C_5|a|, DT^n(x)a \in L \right\} \\ &= \left\{ \begin{pmatrix} v \\ D\tau_n(x)DT^{-n}(x)v+b \end{pmatrix} : v \in L, |b| \leq C_5|DT^{-n}(x)v| \right\}. \end{aligned}$$

And consequently $\mathcal{D}^n(x_1)\mathcal{K} \cap \mathcal{D}^n(x_2)\mathcal{K} \cap L \times \mathbb{R} = \{0\}$ implies that

$$\begin{aligned} &|[(D\tau_n(x_1)DT^{-1}(x_1) - (D\tau_n(x_2)DT^{-1}(x_2))]v| \\ &> C_5|DT^{-n}(x_1)v| + C_5|DT^{-n}(x_2)v|. \end{aligned}$$

□

For all $n \in \mathbb{N}$, let

$$\phi(n) := \sup_{y \in X} \sup_{x_0 \in T^{-n}y} \sum_{\substack{x \in T^{-n}y \\ \mathcal{D}^n(x)\mathcal{K} \not\supset \mathcal{D}^n(x_0)\mathcal{K}}} J_n(x).$$

Let h_ν denote the density of ν (the T -invariant probability measure). It is convenient to introduce the quantity

$$(12) \quad \varphi(n, P, y) := \sum_{\substack{x \in T^{-n}(y) \\ \mathcal{D}^n(x)\mathcal{K} \supset P}} J_n(x) \cdot \frac{f_\nu(x)}{f_\nu(y)}.$$

where $P \subset \mathbb{R}^{d+1}$ is a d -dimensional linear subspace. Let

$$\varphi(n) := \sup_y \sup_P \varphi(n, P, y).$$

The benefit of this definition is that $\varphi(n)$ is submultiplicative, i.e., $\varphi(n+m) \leq \varphi(n)\varphi(m)$ for all $n, m \in \mathbb{N}$; and $\varphi(n) \leq 1$ for all $n \in \mathbb{N}$. In order to prove Proposition 6 it suffices to prove the following lemma.

Lemma 8. *The following statements are equivalent.*

- (i) $\liminf_{n \rightarrow \infty} \phi(n)^{\frac{1}{n}} = 1$;
- (ii) $\lim_{n \rightarrow \infty} \varphi(n)^{\frac{1}{n}} = 1$;
- (iii) *For all $n \in \mathbb{N}$ and $y \in X$ there exists a d -dimensional linear subspace $Q_n(y) \subset \mathcal{K}$ such that $\mathcal{D}^n(x)\mathcal{K} \supset Q_n(y)$ for all y and for all $x \in T^{-n}y$;*
- (iv) *There exists $\theta \in \mathcal{C}^1(X, \mathbb{R})$ such that $\tau = \theta \circ T - \theta + \chi$ where χ is constant on each partition element.*

Proof of (i) \implies (ii). Let $m_2 \in \mathbb{N}$, $n = \lfloor 2^{\frac{\Lambda}{\lambda}} m_2 \rfloor$. Since $\Lambda \geq \lambda$, $n > m_2$. Let $m_1 \in \mathbb{N}_+$ be such that $n = m_1 + m_2$. Let $P_n(x_1) := \mathcal{D}^n(x_1)(\mathbb{R}^d \times \{0\})$. We will first show that $\mathcal{D}^n(x_1)\mathcal{K} \not\supset \mathcal{D}^n(x_2)\mathcal{K}$ implies that $\mathcal{D}^{m_2}(T^{m_1}x_2)\mathcal{K} \supset P_n(x_1)$. Observe that

$$\mathcal{D}^n(x)\mathcal{K} = \left\{ \begin{pmatrix} a \\ D\tau_n(x)DT^{-n}(x)a+b \end{pmatrix} : a \in \mathbb{R}^d, b \in \mathbb{R}, |b| \leq C_5|DT^{-n}(x)a| \right\}.$$

That transversality fails means that $P_n(x_1)$ (being contained in $\mathcal{D}^n(x_1)\mathcal{K}$) is close to the image cone $\mathcal{D}^n(x_2)\mathcal{K}$ by a factor of $C_5 e^{-\lambda n}$. We also know that $\mathcal{D}^{m_2}(T^{m_1}x_2)$ is sufficiently bigger than $\mathcal{D}^n(x_2)\mathcal{K}$ in the sense that

$$\mathcal{D}^{m_2}(T^{m_1}x_2) \supset \left\{ \begin{pmatrix} a \\ b_1 + b_2 \end{pmatrix} : \begin{pmatrix} a \\ b_1 \end{pmatrix} \in \mathcal{D}^n(x_2)\mathcal{K}, |b_2| \leq C_5 e^{-\lambda n} |a| \right\}.$$

To prove this let $a \in \mathbb{R}^d$ and $b_0, b_1, b_2 \in \mathbb{R}$ such that $|b_0| \leq C_5 |DT^{-n}(x_2)a|$, $b_1 = D\tau_n(x_2)DT^{-n}(x_2)a + b_0$ and $|b_2| \leq C_5 e^{-\lambda n} |a|$. It will suffice to prove that

$$|(b_1 + b_2 - D\tau_{m_2}(T^{m_1}x_2)DT^{-m_2}(T^{m_1}x_2))a| \leq C_5 |DT^{-m_2}(T^{m_1}x_2))a|.$$

We estimate

$$\begin{aligned} (13) \quad & |(b_1 + b_2 - D\tau_{m_2}(T^{m_1}x_2)DT^{-m_2}(T^{m_1}x_2))a| \\ &= |(b_0 + b_1 + D\tau_{m_1}(x_2)DT^{-m_1}(x_2))DT^{-m_2}(T^{m_1}x_2)a| \\ &\leq C_5 \left(\frac{1}{2} |DT^{-m_2}(T^{m_1}x_2))a| + 2e^{-\lambda n} |a| \right) \\ &\leq C_5 (2e^{-\lambda n} |a| - \frac{1}{2} |DT^{-m_2}(T^{m_1}x_2))a|) + C_5 |DT^{-m_2}(T^{m_1}x_2))a| \end{aligned}$$

Since $|DT^{-m_2}(T^{m_1}x_2))a| \geq e^{-\Lambda m_2} \geq e^{-\frac{\Lambda}{2}n}$ we see that $\frac{1}{2} |DT^{-m_2}(T^{m_1}x_2))a| \geq 2e^{-\lambda n} |a|$ for n sufficiently large (dependent only on λ and Λ). We therefore conclude that $P_n(x_1) \subset \mathcal{D}^{m_2}(T^{m_1}x_2)$. Suppose that $x_1 \in T^{-n}y$.

$$\begin{aligned} \sum_{\substack{x_2 \in T^{-n}y \\ \mathcal{D}^n(x_2)\mathcal{K} \not\supset \mathcal{D}^n(x_1)\mathcal{K}}} J_n(x_2) &\leq \sum_{\substack{x_2 \in T^{-n}y \\ \mathcal{D}^{m_2}(T^{m_1}x_2)\mathcal{K} \supset P_n(x_1)}} J_{m_2}(T^{m_1}x_2)J_{m_1}(x_2) \\ &\leq \sum_{\substack{x_3 \in T^{-m_2}y \\ \mathcal{D}^{m_2}(x_3)\mathcal{K} \supset P_n(x_1)}} J_{m_2}(x_3) \sum_{x_2 \in T^{-m_1}x_3} J_{m_1}(x_2). \end{aligned}$$

Consequently $\varphi(n) \leq C\phi(m_2(n))$ where $m_2(n) = \lfloor \frac{n\lambda}{2\Lambda} \rfloor$ and $C = \sup_{x,y} \frac{f_\nu(x)}{f_\nu(y)}$. \square

Proof of (ii) \implies (iii). First note that $\lim_{n \rightarrow \infty} \varphi(n)^{\frac{1}{n}} = 1$ implies that $\varphi(n) = 1$ for all n since $\varphi(n)$ is submultiplicative and bounded by 1. Consequently the following statement holds:

(ii') For each n there exists some $y_n \in X$ and some d -dimensional linear subspace $Q_n \subset \mathbb{R}^{d+1}$ such that $\mathcal{D}^n(x)\mathcal{K} \supset Q_n$ for every $x \in T^{-n}(y_n)$.

It remains to prove that this above statement implies the following.

(iii) For all $n \in \mathbb{N}$ and $y \in X$ there exists a d -dimensional linear subspace $Q_n(y) \subset \mathcal{K}$ such that $\mathcal{D}^n(x)\mathcal{K} \supset Q_n(y)$ for all y and for all $x \in T^{-n}y$.

We will prove the contrapositive. Suppose the negation of (ii), i.e., there exists $n_0 \in \mathbb{N}$, $y_0 \in X$, $x_1, x_2 \in T^{-n_0}(y_0)$ such that $\mathcal{D}^{n_0}(x_1)\mathcal{K} \cap \mathcal{D}^{n_0}(x_2)\mathcal{K}$ does not contain a d -dimensional linear subspace. Let $\omega_1, \omega_2 \in \mathcal{P}_{n_0}$ be such that $(x_1 = \ell_{\omega_1}y_0, x_2 = \ell_{\omega_2}y_0)$. These inverses are defined on some neighbourhood Δ containing y_0 and due to the openness related to the cones not intersecting we can assume that $\mathcal{D}^{n_0}(\ell_{\omega_1}(y_0))\mathcal{K} \cap \mathcal{D}^{n_0}(\ell_{\omega_2}(y_0))\mathcal{K}$ does not contain a d -dimensional linear subspace for all $y \in \Delta$ (shrinking Δ as required).

There exists $m_0 \in \mathbb{N}$ and $v \in \mathcal{P}_{m_0}$ such that $\ell_v X \subset \Delta$ (using the covering property of T). Observe that, for all $z \in X$,

$$\mathcal{D}^{n_0+m_0}(\ell_{\omega_1}(\ell_v z))\mathcal{K} \subset \mathcal{D}^{m_0}(\ell_v z)\mathcal{D}^{n_0}(\ell_{\omega_1}y)\mathcal{K}$$

where $y = \ell_v z$ (and similarly for ω_2). This means that for all $z \in X$ there exist $x_1, x_2 \in T^{-(m_0+n_0)}(z)$ such that $\mathcal{D}^{m_0+n_0}(x_1)\mathcal{K} \cap \mathcal{D}^{m_0+n_0}(x_2)\mathcal{K}$ fails to contain a d -dimensional linear subspace and consequently contradicts (i'). \square

Proof of (iii) \implies (iv). Let $(\omega_1, \omega_2, \dots)$ be a sequence of elements of the partition \mathcal{P} . For each $n \in \mathbb{N}$ let $G_n := \ell_{\omega_n} \circ \dots \circ \ell_{\omega_2} \circ \ell_{\omega_1}$. Consider

$$(14) \quad D(\tau_n \circ G_n)(x) = \sum_{k=1}^n D(\tau \circ \ell_{\omega_k})(G_{k-1}x) D G_{k-1}(x)$$

and observe that, by (3) and (1) this series converges uniformly. Moreover this limit is independent of the choice of sequence of inverse branches. This is a consequence of (ii). Observe that

$$\mathcal{D}^n(x)\mathcal{K} = \left\{ \left({}_{D\tau_n(x)DT^{-n}(x)}^v \right) : v \in \mathbb{R}^d, |b| \leq C_5 |DT^{-n}(x)v| \right\}.$$

Therefore, for all $n, y \in X$, then

$$\|D\tau_n(x_1)DT^{-n}(x_1)v - D\tau_n(x_2)DT^{-n}(x_2)v\| \leq 2C_5 \|v\| \lambda^{-n}$$

for all $x_1, x_2 \in T^{-n}y$.

Consequently we can denote by $\Omega(x)$ the limit of (14). It holds that, for all $\omega \in \mathcal{P}$,

$$\Omega(x) = D(\tau \circ \ell_\omega)(x) + \Omega(\ell_\omega x) D\ell_\omega(x).$$

Fix $x_0 \in X$. The series of functions $\sum_{k=1}^\infty (\tau \circ G_n - \tau \circ G_n(x_0))$ is summable in \mathcal{C}^1 . Denote this sum by θ . By construction $\Omega(x) = D\theta(x)$. Consequently $D(\tau + \theta - \theta \circ T) = 0$. \square

Proof of (iv) \implies (i). Let

$$Q(x) := \left\{ \left({}_{D\theta(x)a}^a \right) : a \in \mathbb{R}^d \right\}.$$

Observe that $Q(x) \subset \mathcal{K}$. Since $D\tau_n(x) = D\theta(T^n x)DT^n(x) - D\theta$,

$$\begin{aligned} \mathcal{D}(x)^n Q(x) &= \left\{ \left({}_{D\tau_n(x)}^{DT^n(x) \ 0} \right) \left({}_{D\theta(x)a}^a \right) : a \in \mathbb{R}^d \right\} \\ &= \left\{ \left({}_{D\theta(T^n x)DT^n(x)a}^{DT^n(x)a} \right) : a \in \mathbb{R}^d \right\} = Q(T^n x). \end{aligned}$$

This means that for all $y \in X$ then $\mathcal{D}(x)^n \mathcal{K} \supset Q(y)$ for all $x \in T^{-n}y$. \square

3.4. Oscillatory Cancellation. In this subsection we take advantage of the geometric property established above and estimate the resultant cancellations. The following estimate concerns the case when f is more or less constant on a scale of $|b|^{-1}$. The argument will depend on the following choice of constants (chosen conveniently but not optimally)

$$\beta_1 := \frac{2}{\lambda}, \quad \beta_2 := \frac{\alpha}{8\Lambda}, \quad q := \frac{\alpha\lambda}{2}.$$

Let $n_1 = \lfloor \beta_1 \ln |b| \rfloor$, $n_2 = \lfloor \beta_2 \ln |b| \rfloor$ and $n := n_1 + n_2$, $\beta := \beta_1 + \beta_2$. The first n_1 iterates will be so that the dynamics evenly spreads the function f across the space X . Then n_2 iterates will be to see the oscillatory cancellations.

Proposition 9. *There exists $\xi > 0$, $b_0 > 1$ such that, for all $z = a + ib$, $a \in (-\sigma, \sigma)$, $|b| > b_0$, $n = \lfloor \beta \ln |b| \rfloor$, for all $f \in \mathcal{C}^\alpha(X)$, $|f|_{\mathcal{C}^\alpha(X)} \leq e^{qn} |b|^\alpha \|f\|_{L^\infty(X)}$,*

$$\|\mathcal{L}_z^n f\|_{L^1(X)} \leq e^{-\xi n} \|f\|_{L^\infty(X)}.$$

The proof follows after several lemmas. It is convenient to localize in space using a partition of unity. Using that X is a bounded subset of \mathbb{R}^d and that $\partial\omega$ (the boundary) is smooth for each $\omega \in \mathcal{P}$ we have the following.

Lemma 10 ([17, Footnote 15]). *There exists $C_{10}, C_{11}, r_0 > 0$ such that, for all $r \in (0, r_0)$ there exists a set of points $\{x_p\}_{p=1}^{N_r}$ and a C^1 partition of unity $\{\rho_p\}_{p=1}^{N_r}$ of X (i.e., $\sum_p \rho \equiv 1$, $\rho_p \in C^1(\mathcal{M}, [0, 1])$) with the following properties.*

- $N_r \leq C_{10}r^{-d}$;

For each p ,

- $\rho_p(x) = 1$ for all $x \in B(x_p, r)$;
- $\text{Supp}(\rho_p) \subset B(x_p, C_{10}r)$;
- $\|\rho_p\|_{C^1} \leq C_{10}r^{-1}$;

And, letting $\mathcal{R}_\partial := \{p \in \{1, \dots, N_r\} : B(x_p, C_{10}r) \cap \partial\omega \text{ for some } \omega \in \mathcal{P}\}$,

- $\#\mathcal{R}_\partial \leq C_{11}r^{-(d-1)}$.

At each different point of X we have a direction in which we see cancelations. The purpose of the partition of unity is to consider the direction as locally constant. We choose $r = r(b) = |b|^{-\frac{1}{2}}$. Take $f \in \mathcal{C}^\alpha(X)$. Using Jensen's inequality

$$\begin{aligned} \|\mathcal{L}_z^n f\|_{L^1(X)} &= \int_X \left| \sum_{\omega \in \mathcal{P}_n} (J_n \cdot f \cdot e^{-z\tau_n}) \circ \ell_\omega(x) \cdot \mathbf{1}_{T^n\omega}(x) \right| dx \\ &= \sum_{p=1}^{N_r} \int \left| \sum_{\omega \in \mathcal{P}_n} \rho_p \cdot (J_n \cdot f \cdot e^{-z\tau_n}) \circ \ell_\omega(x) \cdot \mathbf{1}_{T^n\omega}(x) \right| dx \\ &\leq \left(\sum_{p \notin \mathcal{R}_\partial} \sum_{\omega, v \in \mathcal{P}_n} \left| \int_{T^n\omega \cap T^n v} (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_v \cdot e^{ib\theta_{\omega, v}})(x) dx \right| \right)^{\frac{1}{2}} \\ &\quad + e^{\sigma n} \|f\|_{L^\infty} \sum_{\omega \in \mathcal{P}_n} \|J_n\|_{L^\infty(\omega)} \sum_{p \in \mathcal{R}_\partial} \int_{T^n\omega} \rho_p dx \end{aligned}$$

where $K := (J_n \cdot f \cdot e^{-a\tau_n})$ and $\theta_{\omega, v} := \tau_n \circ \ell_\omega - \tau_n \circ \ell_v$. Using Lemma 3 and Lemma 10 the final term of the above is bounded by

$$(15) \quad C_{10}^d C_7 r e^{\sigma n} \|f\|_{L^\infty(X)} \leq C_{10}^d C_7 e^{-(\frac{1}{2\beta} - \sigma)n} \|f\|_{L^\infty(X)}.$$

It remains to estimate the other term. We estimate separately the set

$$\mathcal{Q}_{n,p,\omega} := \{v \in \mathcal{P}_n : \mathcal{D}^{n_2}(T^{n_1}\ell_v x_p) \cap \mathcal{D}^{n_2}(T^{n_1}\ell_\omega x_p)\}$$

and the set of v where this is not the case. In the second case we see oscillatory cancellations.

Lemma 11. *There exists $C_{12} > 0$ such that*

$$|K \circ \ell_\omega(x) - K \circ \ell_\omega(y)| \leq e^{\sigma n} \|J_n\|_{L^\infty(\omega)} (C_{12} \|f\|_{L^\infty(\omega)} + |f|_{\mathcal{C}^\alpha(\omega)} e^{-\alpha\lambda n}) d(x, y)^\alpha$$

for all $n \in \mathbb{N}$, $\omega \in \mathcal{P}_n$, $x, y \in T^n\omega$.

Proof. Since $K \circ \ell_\omega(x) = (J_n \cdot f \cdot e^{-a\tau_n}) \circ \ell_\omega(x)$, for all $x, y \in T^n\omega$,

$$\begin{aligned} K \circ \ell_\omega(x) - K \circ \ell_\omega(y) &= (e^{-a\tau_n(\ell_\omega x)} - e^{-a\tau_n(\ell_\omega y)}) f(\ell_\omega x) \cdot J_n(\ell_\omega x) \\ &\quad + e^{-a\tau_n(\ell_\omega y)} f(\ell_\omega y) (J_n(\ell_\omega x) - J_n(\ell_\omega y)) \\ &\quad + e^{-a\tau_n(\ell_\omega y)} (f(\ell_\omega x) - f(\ell_\omega y)) \cdot J_n(\ell_\omega y). \end{aligned}$$

Using the estimates of Lemma 1, Lemma 2 and (1),

$$\begin{aligned} |K \circ \ell_\omega(x) - K \circ \ell_\omega(y)| &\leq e^{\sigma n} \|J_n\|_{L^\infty(\omega)} \left(\left(\sigma \frac{C_5}{2} + C_6 \right) \|f\|_{L^\infty(\omega)} + |f|_{\mathcal{C}^\alpha(\omega)} e^{-\alpha\lambda n} \right) d(x, y)^\alpha \end{aligned}$$

The lemma follows from choosing $C_{12} := C_6 + \sigma \frac{C_5}{2}$. \square

Lemma 12. *There exists $C_{13} > 0$ such that, for all $n \in \mathbb{N}$, $\omega, v \in \mathcal{P}_n$,*

$$\|D\theta_{\omega,v}\|_{\alpha} \leq C_{13}.$$

Proof. $D(\tau_n \circ \ell_{\omega})(x) = \sum_{k=0}^{n-1} D\tau(h_k x) Dh_k(x)$ where $h_k := T^k \circ \ell_{\omega}$. So

$$\begin{aligned} \|D(\tau_n \circ g)(x) - D(\tau_n \circ g)(y)\| &\leq \sum_{k=0}^{n-1} \|D\tau\|_{\alpha} d(h_k x, h_k y)^{\alpha} \\ &\leq \|D\tau\|_{\alpha} \sum_{k=0}^{n-1} e^{-\lambda n \alpha} d(x, y)^{\alpha}. \end{aligned}$$

And so $\|D\theta_{g,h}\|_{\alpha} \leq 2 \|D\tau\|_{\alpha} \sum_{k=0}^{\infty} e^{-\lambda n \alpha}$. \square

Lemma 13. *Suppose the setting of Proposition 9. There exists $C_{14} > 0$ such that*

$$\begin{aligned} (16) \quad &\left(\sum_{p \notin \mathcal{R}_{\partial}} \sum_{\omega \in \mathcal{P}_n} \sum_{v \in \mathcal{Q}_{n,p,\omega}} \left| \int (\rho_p \cdot K \circ \ell_{\omega} \cdot K \circ \ell_v \cdot e^{ib\theta_{\omega,v}})(x) dx \right| \right)^{\frac{1}{2}} \\ &\leq C_{14} e^{-(\frac{\gamma\beta_2}{2\beta} - \sigma)n} \|f\|_{L^{\infty}}. \end{aligned}$$

Proof. Fixing for the moment $p \in \mathcal{R}_{\partial}$ and $\omega \in \mathcal{P}_n$ we want to perform the sum over v .

$$\begin{aligned} (17) \quad &\sum_{v \in \mathcal{Q}_{n,p,\omega}} \left| \int_{T^n \omega \cap T^n v} (\rho_p \cdot K \circ \ell_v \cdot K \circ \ell_{\omega} \cdot e^{ib\theta_{\omega,v}})(x) dx \right| \\ &\leq \left(\sum_{v \in \mathcal{Q}_{n,p,\omega}} \|J_n\|_{L^{\infty}(v)} \right) e^{2\sigma n} \|J_n\|_{L^{\infty}(\omega)} \|f\|_{L^{\infty}}^2 \|\rho_p\|_{L^1}. \end{aligned}$$

Observe that

$$\sum_{v \in \mathcal{Q}_{n,p,\omega}} \|J_n\|_{L^{\infty}(v)} \leq \left(\sum_{v_1 \in \mathcal{P}_{n_1}} \|J_n\|_{L^{\infty}(v_1)} \right) \left(\sum_{v_2} \|J_n\|_{L^{\infty}(v_2)} \right)$$

where the second sum is over the set of $v_2 \in \mathcal{P}_{n_2}$ which satisfying

$$\mathcal{D}^{n_2}(T^{n_1} \ell_{v_2} x_p) \cap \mathcal{D}^{n_2}(T^{n_1} \ell_{\omega} x_p)$$

Consequently, applying the estimate of Proposition 6, the term in (17) is bounded by

$$C_9 C_7 e^{-\gamma n_2} e^{2\sigma n} \|J_n\|_{L^{\infty}(\omega)} \|f\|_{L^{\infty}}^2 \|\rho_p\|_{L^1}.$$

Using again Lemma 3, $\sum_{\omega \in \mathcal{P}_n} \|J_n\|_{L^{\infty}(\omega)} \leq C_7$ and we sum over p . \square

Now we turn our attention to the $v \in \mathcal{P}_n$ where we observe oscillatory cancellations. The crucial technical part of the estimate is the following oscillatory integral bound.

Lemma 14. *Suppose that $J \subset [0, 1]$ is an interval, $k \in \mathcal{C}^{\alpha}(J)$, $\theta \in \mathcal{C}^{1+\alpha}(J)$, $|\theta'| \geq \kappa > 0$, $|b| > 1$, $k \in \mathcal{C}^{\alpha}(J)$. Then*

$$\left| \int_J e^{ib\theta(x)} k(x) dx \right| \leq \frac{C}{\kappa^2 |b|^{\alpha}} \|k\|_{\mathcal{C}^{\alpha}(J)}.$$

where $C = (\|\theta'\|_{L^{\infty}(X)} + 6)(1 + |\theta'|_{\mathcal{C}^{\alpha}(X)})$.

Proof. We assume that $b > 1$, the other case being identical. We also assume without loss of generality that $\theta' \geq \kappa$ otherwise we can exchange $-\theta$ for θ . Since $\frac{k}{\theta'}$ is α -Hölder there exists⁸ $g_b \in \mathcal{C}^1(J, \mathbb{R})$ such that

$$\|g_b - \frac{k}{\theta'}\|_\infty \leq b^{-\alpha} \left| \frac{k}{\theta'} \right|_\alpha, \quad \|g'_b\|_\infty \leq 2b^{1-\alpha} \left| \frac{k}{\theta'} \right|_\alpha.$$

Changing variables, $y = \theta(x)$,

$$\begin{aligned} \int_J k(x) \cdot e^{ib\theta(x)} dx &= \int_{\theta(J)} \frac{k}{\theta'} \circ \theta^{-1}(y) e^{iby} dy \\ &= \int_{\theta(J)} g_b \circ \theta^{-1}(y) e^{iby} dy + \int_{\theta(J)} \left(\frac{k}{\theta'} - g_b \right) \circ \theta^{-1}(y) e^{iby} dy. \end{aligned}$$

Integrating by parts

$$\begin{aligned} \int_{\theta(J)} g_b \circ \theta^{-1}(y) e^{iby} dy &= -\frac{i}{b} [g_b \circ \theta^{-1}(y) e^{iby}]_{\theta(J)} + \frac{i}{b} \int_{\theta(J)} \frac{g'_b}{\theta'} \circ \theta^{-1}(y) e^{iby} dy \\ &= -\frac{i}{b} [g_b e^{ib\theta}]_J + \frac{i}{b} \int_J g'_b(x) e^{ib\theta(x)} dx. \end{aligned}$$

Combining these estimates

$$\left| \int_J e^{ib\theta(x)} k(x) dx \right| \leq \left(\frac{\|\theta'\|_\infty |J|}{b^\alpha} + \frac{2}{b^{1+\alpha}} + \frac{2|J|}{b^\alpha} \right) \left| \frac{k}{\theta'} \right|_\alpha + \frac{2\|k\|_\infty}{b\kappa}.$$

To finish we observe that

$$\begin{aligned} \left| \frac{k}{\theta'}(x) - \frac{k}{\theta'}(y) \right| &= \left| \frac{k(x) - k(y)}{\theta'(x)} + \frac{k(y)(\theta'(y) - \theta'(x))}{\theta'(x)\theta'(y)} \right| \\ &\leq \left(\frac{|k|_\alpha}{\kappa} + \frac{\|k\|_\infty |\theta'|_\alpha}{\kappa^2} \right) |x - y|^\alpha. \end{aligned}$$

□

Lemma 15. *Suppose the setting of Proposition 9. There exists $C_{15} > 0$ such that*

$$\begin{aligned} (18) \quad & \left(\sum_{p \notin \mathcal{R}_\theta} \sum_{\omega \in \mathcal{P}_n} \sum_{v \in \mathcal{P}_n \setminus \mathcal{Q}_{n,p,\omega}} \left| \int (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_v \cdot e^{ib\theta_{\omega,v}})(x) dx \right| \right)^{\frac{1}{2}} \\ & \leq C_{15} |b|^{-\frac{\alpha}{4}} e^{(\frac{\Lambda\beta_2}{\beta} + \sigma)n} \|f\|_{L^\infty} \leq C_{15} e^{-(\frac{\alpha}{8\beta} - \sigma)n} \|f\|_{L^\infty}. \end{aligned}$$

Proof. Fixing for the moment p and ω we want to perform the sum over v . I.e., we estimate

$$\sum_{v \in \mathcal{P}_n \setminus \mathcal{Q}_{n,p,\omega}} \left| \int_{T^n \omega \cap T^n v} (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_v \cdot e^{ib\theta_{\omega,v}})(x) dx \right|$$

Since $\mathcal{D}^{n_2}(T^{n_1} \ell_v x_p) \cap \mathcal{D}^{n_2}(T^{n_1} \ell_\omega x_p)$ there exists (Lemma 7) a 1-dimensional linear subspace $L \subset \mathbb{R}^d$ such that, for all $v \in L$,

$$|D(\tau_{n_2} \circ T^{n_1} \circ \ell_v)(x_p)v - D(\tau_{n_2} \circ T^{n_1} \circ \ell_\omega)(x_p)v| > C_5 |D(T^{n_1} \circ \ell_v)(x_p)v|.$$

By Lemma 1

$$|D(\tau_{n_1} \circ \ell_v)(x_p)v| \leq \frac{C_5}{2} |D(T^{n_1} \circ \ell_v)(x_p)v|.$$

Consequently

$$|D(\tau_n \circ \ell_v)(x_p)v - D(\tau_n \circ \ell_\omega)(x_p)v| > \frac{C_5}{2} (|D(T^{n_1} \circ \ell_v)(x_p)v| + |D(T^{n_1} \circ \ell_\omega)(x_p)v|).$$

⁸ Take a mollifier $\rho \in \mathcal{C}^1(\mathbb{R}, [0, 1])$ such that $\text{Supp}(\rho) \subset (-1, 1)$, $\int \rho = 1$, $\int |\rho'| \leq 2$. Define

$$g_b(x) := \int \rho_b(x - y) \frac{k}{\theta'}(y) dy$$

where $\rho_b(z) := b\rho(bz)$. Observe that $g_b(x) - \frac{k}{\theta'}(x) = \int \rho_b(x - y) \left[\frac{k}{\theta'}(y) - \frac{k}{\theta'}(x) \right] dy$, that $g'_b(x) = \int \rho'_b(x - y) \left[\frac{k}{\theta'}(y) - \frac{k}{\theta'}(x) \right] dy$, that $\int |\rho_b| = 1$ and that $\int |\rho'_b| \leq 2b$.

Rotating and translating the axis, we choose an orthogonal coordinate system (y_1, y_2, \dots, y_d) such that y_1 corresponds to L and such that $x_p = (0, \dots, 0)$. We have

$$\left| \frac{\partial \theta_{\omega, v}}{\partial y_1} \right| (0, \dots, 0) = \left| \frac{\partial(\tau_n \circ \ell_v)}{\partial y_1} - \frac{\partial(\tau_n \circ \ell_\omega)}{\partial y_1} \right| (0, \dots, 0) \geq C_5 e^{-\Lambda n_2}.$$

Since $r > 0$ is sufficiently small the transversality holds along this direction for the entire ball (using Lemma 12). It suffices to show that $C_{13} r_b^\alpha \leq \frac{C_5}{2} e^{-\Lambda n_2}$ since $\|D\theta_{\omega, v}\|_\alpha \leq C_{13}$. This is equivalent to requiring $\exp(-[\frac{\alpha}{2\beta_2} - \Lambda]n_2) \leq \frac{C_5}{2C_{13}}$ which holds for $|b|$ sufficiently large since β_2 was chosen such that $\beta_2 \leq \frac{\alpha}{2\Lambda}$. Here b_0 is chosen sufficiently large. We have

$$\left| \frac{\partial \theta_{\omega, v}}{\partial y_1} \right| (y_1, \dots, y_d) = \left| \frac{\partial(\tau_n \circ \ell_v)}{\partial y_1} - \frac{\partial(\tau_n \circ \ell_\omega)}{\partial y_1} \right| (y_1, \dots, y_d) \geq \frac{C_5}{2} e^{-\Lambda n_2}$$

for all $(y_1, \dots, y_d) \in B_{r_b}(0)$. To proceed we must estimate the Hölder norm of $\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_v$. By Lemma 11, since $|f|_{C^\alpha(\omega)} \leq e^{(q+\frac{\alpha}{\beta})n} \|f\|_{L^\infty(\omega)}$ and $q + \frac{\alpha}{\beta} \leq \alpha(\frac{\lambda}{2} + \frac{1}{\beta_1}) = \alpha\lambda$ (for some $C > 0$),

$$|K \circ \ell_\omega(x) - K \circ \ell_\omega(y)| \leq C e^{\sigma n} \|J_n\|_{L^\infty(\omega)} \|f\|_{L^\infty(\omega)} d(x, y)^\alpha$$

Consequently, using Lemma 11 and Lemma 10,

$$|\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_v|_{C^\alpha(T^n \omega)} \leq C (1 + r^{-\alpha}) e^{an} \|J_n\|_{L^\infty(\omega)} \|f\|_{L^\infty(\omega)}.$$

Using the estimate of Lemma 14, for (y_2, \dots, y_d) fixed,

$$\begin{aligned} \left| \int_{-r}^r (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_v \cdot e^{ib\theta_{\omega, v}})(y_1, \dots, y_d) dy_1 \right| \\ \leq C r^{-\alpha} e^{2\Lambda n_2} |b|^{-\alpha} e^{2\sigma n} \|J_n\|_{L^\infty(\omega)} \|J_n\|_{L^\infty(v)} \|f\|_{L^\infty}^2. \end{aligned}$$

If $d = 1$ we are done, otherwise we integrate over the other directions. We also recall that $r = |b|^{-\frac{1}{2}}$.

$$\begin{aligned} \left| \int_{T^n \omega \cap T^n v} (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_v \cdot e^{ib\theta_{\omega, v}})(x) dx \right| \\ \leq C |b|^{-\frac{\alpha}{2}} e^{2\Lambda n_2 + 2\sigma n} \|J_n\|_{L^\infty(\omega)} \|J_n\|_{L^\infty(v)} \|f\|_{L^\infty}^2. \end{aligned}$$

Using Lemma 3 we sum over ω and v to obtain the estimate. \square

Proof of Proposition 9. The estimates from (15), Lemma 15 and Lemma 13 imply that, for some $C > 0$,

$$\|\mathcal{L}_z^n f\|_{L^1(X)} \leq C \left(e^{-(\frac{\gamma\beta_2}{2\beta} - \sigma)n} + e^{-(\frac{\alpha}{8\beta} - \sigma)n} + e^{-(\frac{1}{2\beta} - \sigma)n} \right) \|f\|_{L^\infty(X)}.$$

Here we insure that $\sigma > 0$ is sufficiently small, dependent only on the system. \square

Proposition 16. *There exists $\zeta, b_0, B > 0$ such that, for all $z = a + ib$, $a \geq -\sigma$, $|b| \geq b_0$, $n \geq \lfloor B \ln |b| \rfloor$*

$$\|\mathcal{L}_z^n\|_{(b)} \leq e^{-\zeta n}.$$

Proof. First consider $n = \beta \ln |b|$. We will estimate this quantity in two separate cases. Firstly we consider the case when

$$\|f\|_{L^\infty(X)} \leq e^{-qn} \|f\|_{(b)}.$$

We apply Lemma 5:

$$\|\mathcal{L}_z^n f\|_{(b)} \leq C_8 e^{\sigma n} \left(e^{-\lambda n} \|f\|_{(b)} + \|f\|_{L^\infty(X)} \right) \leq C e^{\sigma n} (e^{-qn} + e^{-\lambda n}) \|f\|_{(b)}$$

It remains to consider the case when $\|f\|_{L^\infty} \geq e^{-qn} \|f\|_{(b)}$. This means that $|f|_{\mathcal{C}^\alpha(X)} \leq e^{qn} |b|^\alpha \|f\|_{L^\infty(X)}$. Observe⁹ that there exists $C, \epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$,

$$(19) \quad \|f\|_{L^\infty(X)} \leq C\epsilon^{-d} \|f\|_{L^1(X)} + \epsilon^\alpha |f|_{\mathcal{C}^\alpha(X)}.$$

Here we choose $\epsilon = e^{-\frac{\xi}{2d}n}$. Applying Lemma 5 twice

$$\|\mathcal{L}_z^{2n} f\|_{(b)} \leq C_8 e^{2\sigma n} e^{-\lambda n} \|f\|_{(b)} + C_8 e^{\sigma n} \|\mathcal{L}_z^n f\|_{L^\infty(X)}.$$

Using also the above estimate (19)

$$\|\mathcal{L}_z^{2n} f\|_{(b)} \leq \left(C_8 e^{-(\lambda-2\sigma)n} + e^{-\frac{\alpha\xi}{2d}n} \right) \|f\|_{(b)} + C_8 e^{(\sigma+\frac{\xi}{2})n} \|\mathcal{L}_z^n f\|_{L^\infty(X)}.$$

The estimate of Proposition 9 means that

$$\|\mathcal{L}_z^{2n} f\|_{(b)} \leq \left(C_8 e^{-(\lambda-2\sigma)n} + e^{-\frac{\alpha\xi}{2d}n} \right) \|f\|_{(b)} + C_8 e^{-(\frac{\xi}{2}-\sigma)n} \|f\|_{(b)}.$$

Again we ensure that $\sigma > 0$ is sufficiently small. We have obtained the estimate $\|\mathcal{L}_z^n f\|_{(b)} \leq e^{-\zeta n}$ when $n = \lfloor \beta \ln |b| \rfloor$. Iterating this estimate and choosing $B > 0$ sufficiently large concludes the proof. \square

3.5. Rate of Mixing. The required conclusion of exponential mixing follows in a standard way (for example [3, §2.7] or [6, §7.5]) from the estimate of Proposition 16. In the first cited reference the \mathcal{C}^1 norm is used whilst in our case the \mathcal{C}^α norm is used but the same argument holds since it depends on the spectral properties of the twisted transfer operator and the norm estimate (Proposition 16) and these are identical in the present case. In the second cited reference the \mathcal{C}^α norm is used but for functions of the interval and not the higher dimensional situation of the present work. Again the argument depends only on the spectral properties of the operator. This completes the proof of Theorem 3.

REFERENCES

- [1] D.V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.* (90), 1967 (A.M.S. translation, 1969).
- [2] V. Araújo, O. Butterley, and P. Varandas. Open sets of Axiom A flows with exponentially mixing attractors. *Proc. Amer. Math. Soc.*, 144(7):2971–2984, 2016.
- [3] V. Araújo and I. Melbourne. Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor. To appear in *Annales Henri Poincaré*. ArXiv:1504.04316.
- [4] V. Araújo and I. Melbourne. Existence and smoothness of the stable foliation for sectional hyperbolic attractors. ArXiv:1604.06924.
- [5] V. Araújo and P. Varandas. Robust exponential decay of correlations for singular-flows. *Comm. Math. Phys.*, 311(1):215–246, 2012 (Errata: 341(2):729–731, 2016).
- [6] A. Avila, S. Gouëzel, and J.-C. Yoccoz. Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.*, 104:143–211, 2006.
- [7] V. Baladi and B. Vallée. Exponential decay of correlations for surface semi-flows without finite Markov partitions. *Proc. Amer. Math. Soc.*, 133(3):865–874, 2005.
- [8] R. Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.*, 95:429–460, 1973.
- [9] R. Bowen. Mixing Anosov Flows. *Topology*, 15(1):77–79, 1976.
- [10] O. Butterley and I. Melbourne. Disintegration of invariant measures for hyperbolic skew products. To appear in *Israel J. Math.*, 2015.
- [11] N.I. Chernov. Markov approximations and decay of correlations for Anosov flows. *Ann. of Math. (2)*, 147:269–324, 1998.

⁹Partition X into connected subsets which each contain a ball of volume ϵ^d . Using the smoothness of the boundary of X we can ensure that, for $\epsilon > 0$ sufficiently small, the diameter of each set is not greater than $C\epsilon$ for some fixed $C > 0$. We may also ensure that each subset is contained within some $\omega \subset \mathcal{P}$. There exists some point x in each of the balls such that $f(x) \leq \|f\|_{L^1} \epsilon^{-d}$. If this were not the case there would exist at least one subset in which $f(x) > \epsilon^{-d} \|f\|_{L^1}$ for all x in that ball and there would be a contradiction. Therefore $\|f\|_{L^\infty} \leq \epsilon^{-d} \|f\|_{L^1(X)} + C\epsilon |f|_{\mathcal{C}^\alpha(X)}$.

- [12] D. Dolgopyat. On decay of correlations in Anosov flows. *Ann. of Math. (2)*, 147(2):357–390, 1998.
- [13] M. Field, I. Melbourne, and A. Török. Stability of mixing and rapid mixing for hyperbolic flows. *Ann. of Math. (2)*, 166(1):269–291, 2007.
- [14] J. Franks and R. Williams. Anomalous Anosov flows. In *Global theory of dynamical systems*, pages 158–174, *Lect. Notes in Math.*, 819. Springer, Berlin, 1980.
- [15] B. Hasselblatt and A. Wilkinson. Prevalence of non-Lipschitz Anosov foliations. *Ergod. Th. & Dynam. Sys.*, 19(3):643–656, 1999.
- [16] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds*, vol. 583 of *Lect. Notes in Math.* Springer Verlag, New York, 1977.
- [17] C. Liverani. On contact Anosov flows. *Ann. Math. (2)*, 159:1275–1312, 2004.
- [18] J.F. Plante. Anosov flows. *Amer. J. Math.*, 94:729–754, 1972.
- [19] M. Tsujii. Exponential mixing for generic volume-preserving Anosov flows in dimension three. arXiv:1601.00063.
- [20] A. Verjovsky. Codimension one Anosov flows. *Bol. Soc. Mat. Mexicana*, 19(2):49–77, 1974.

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